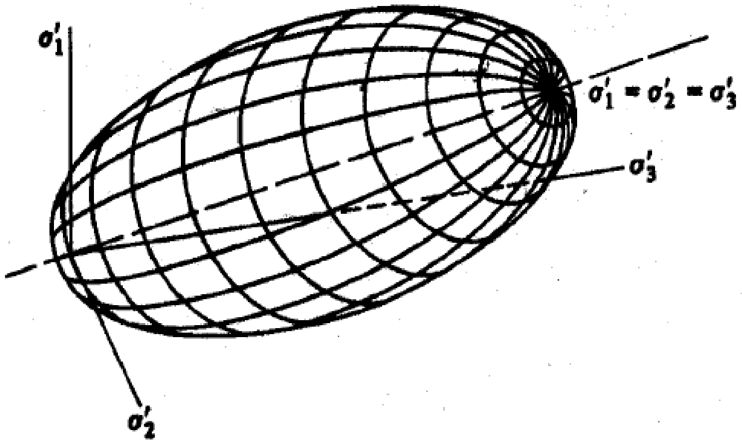

Geomechanics

LECTURE 5

HARDENING ELASTO-PLASTICITY

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Laboratory of soil mechanics – Fall 2025

06.10.2025

Access the QUIZ

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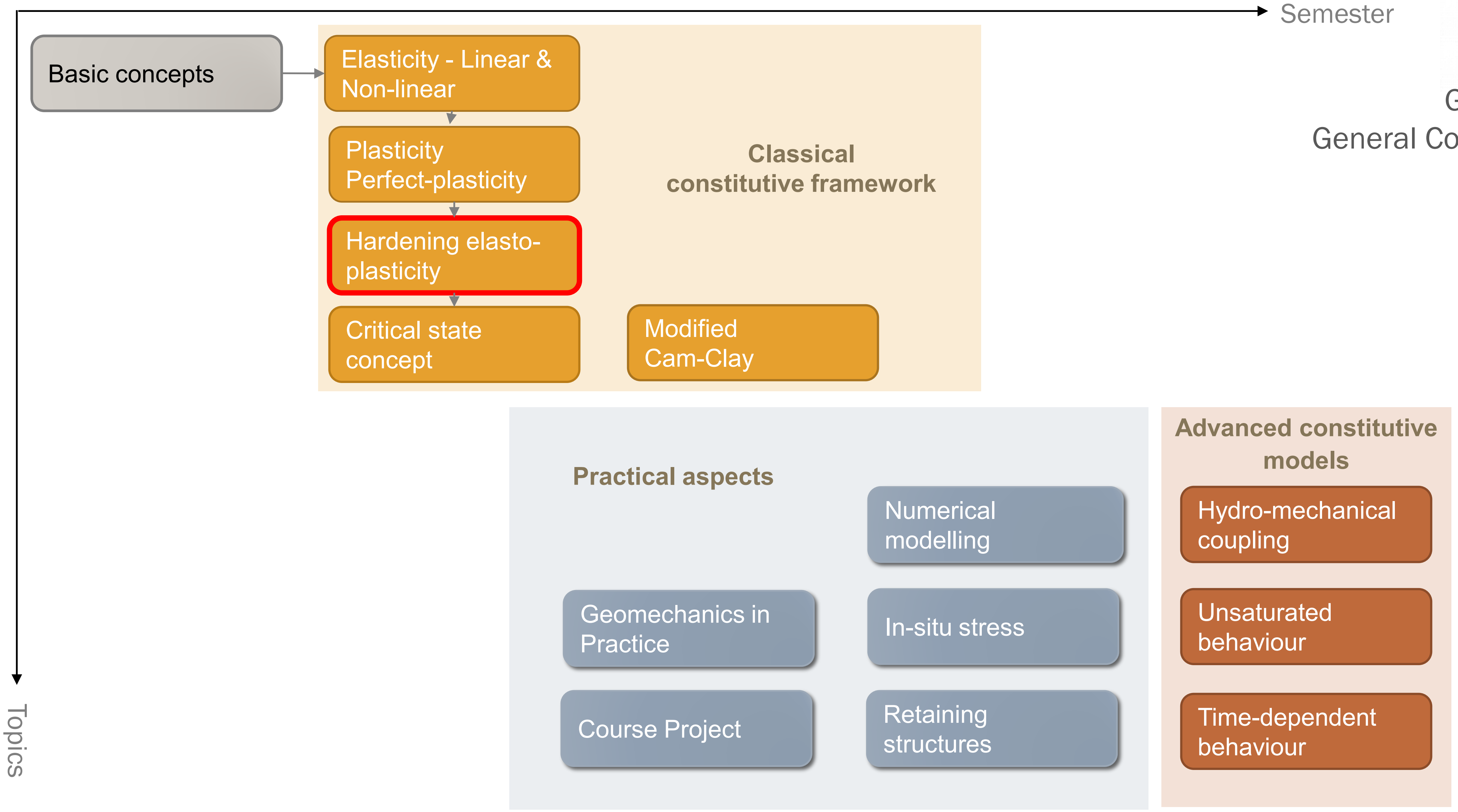
Geomechanics
General Course overview



Content

- Post Yield behaviour and plastic strain
- Hardening behaviour
- Consistency condition
- Hardening elasto-plastic stress-strain relationship
- Extended Mohr-Coulomb model
- Conclusion

Post Yield behaviour and plastic strain

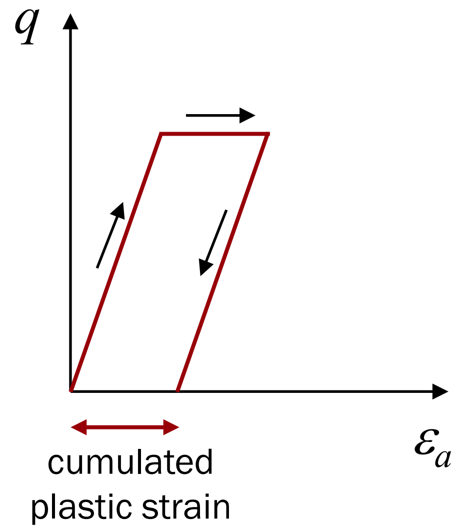
ELASTIC DOMAIN AND YIELD FUNCTION

PLASTIC DEFORMATION

Perfect plasticity vs hardening plasticity

- Perfect plasticity reproduces the accumulation of irreversible deformations, but it assumes yielding = failure.

e.g. elastic-perfectly plastic MC response: occurrence of plastic deformation (**yielding**) when the material reaches the available strength (**failure**).

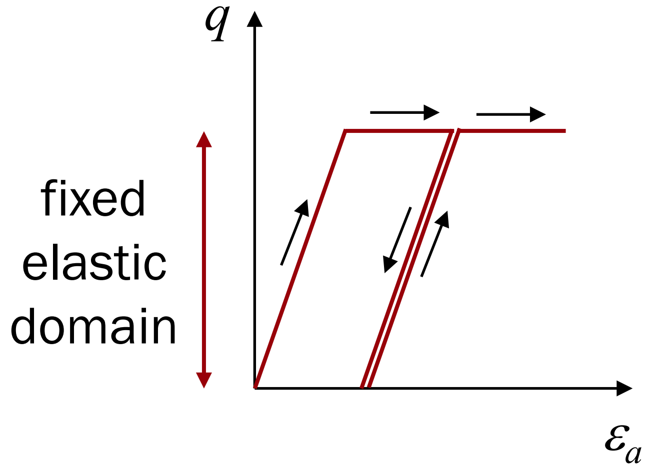


Perfect plasticity vs hardening plasticity

- In perfect plasticity the yield function is fixed, $F = F(\sigma_i, p_k)$

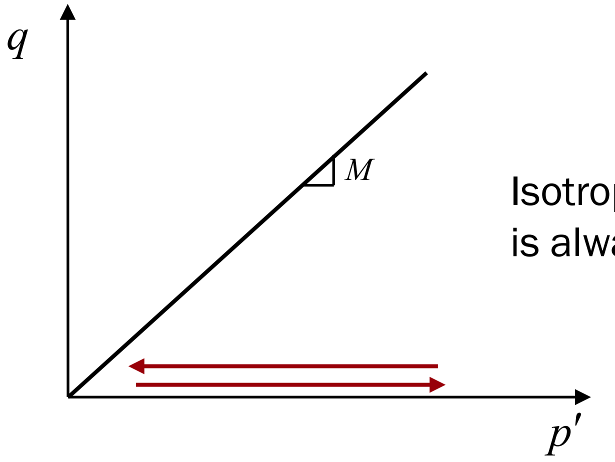
i.e. perfect plasticity doesn't account for the evolution of the elastic domain in shearing.

Elastic-perfectly plastic MC response

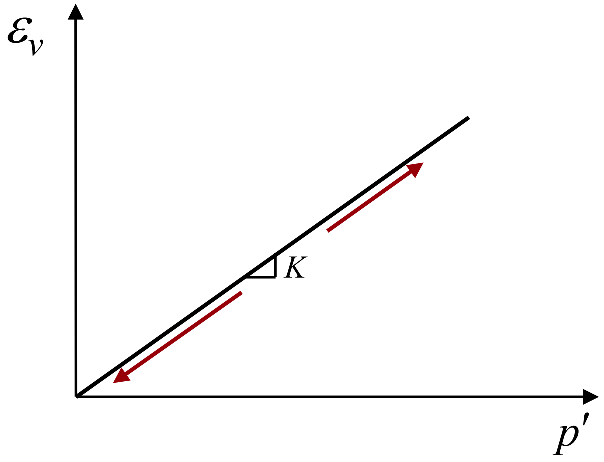


Perfect plasticity vs hardening plasticity

- Other problems of MC elasto-perfectly plastic model:



Isotropic loading/unloading
is always reversible

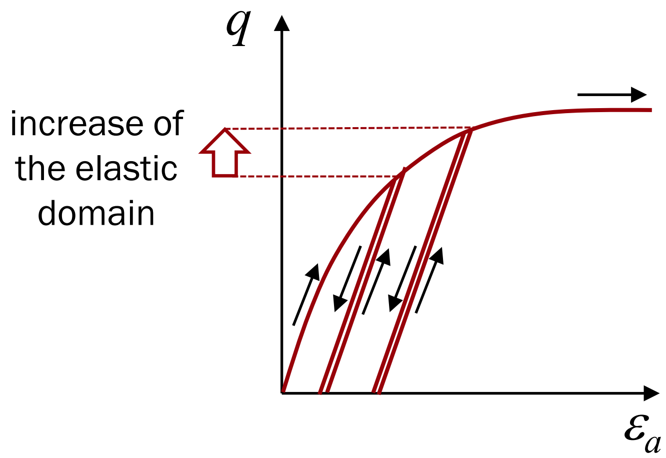


$$\delta\varepsilon_v = \frac{\delta p'}{K}$$

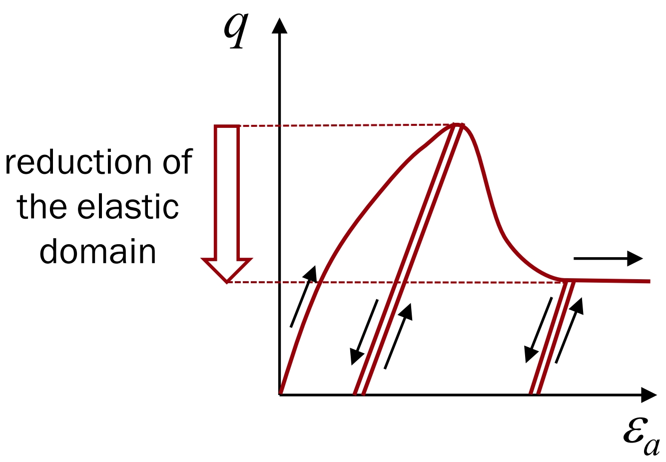
- It is a problem when we are interested in deformation

Perfect plasticity vs hardening plasticity

- The evolution of the elastic domain with loading is a characteristic of geomaterials.



Hardening: expansion of the elastic domain

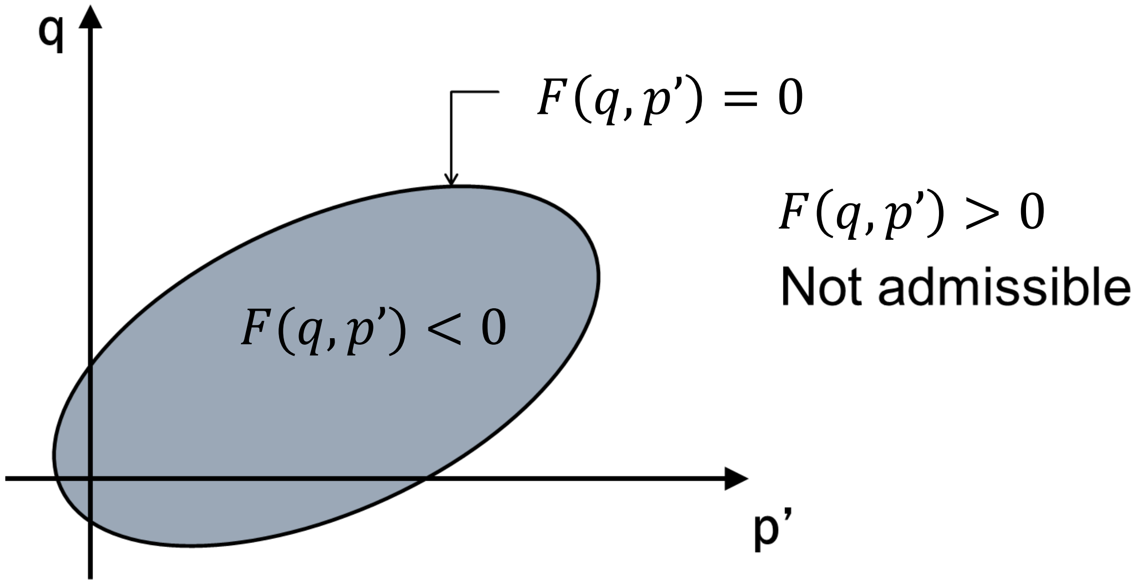


Softening: contraction of the elastic domain

- The difference is related (for the same soil) to OCR (clays) and Relative density (sands).

Elastic domain and yield function

- **Pre-yield behavior**
  Reversible deformation
  Elasticity
- **Yielding**
  Limit between reversible and irreversible behavior
  Yield criteria
- **Post-yield behavior**
  Irreversible deformation
  Full elasto-plastic stress-strain relationship



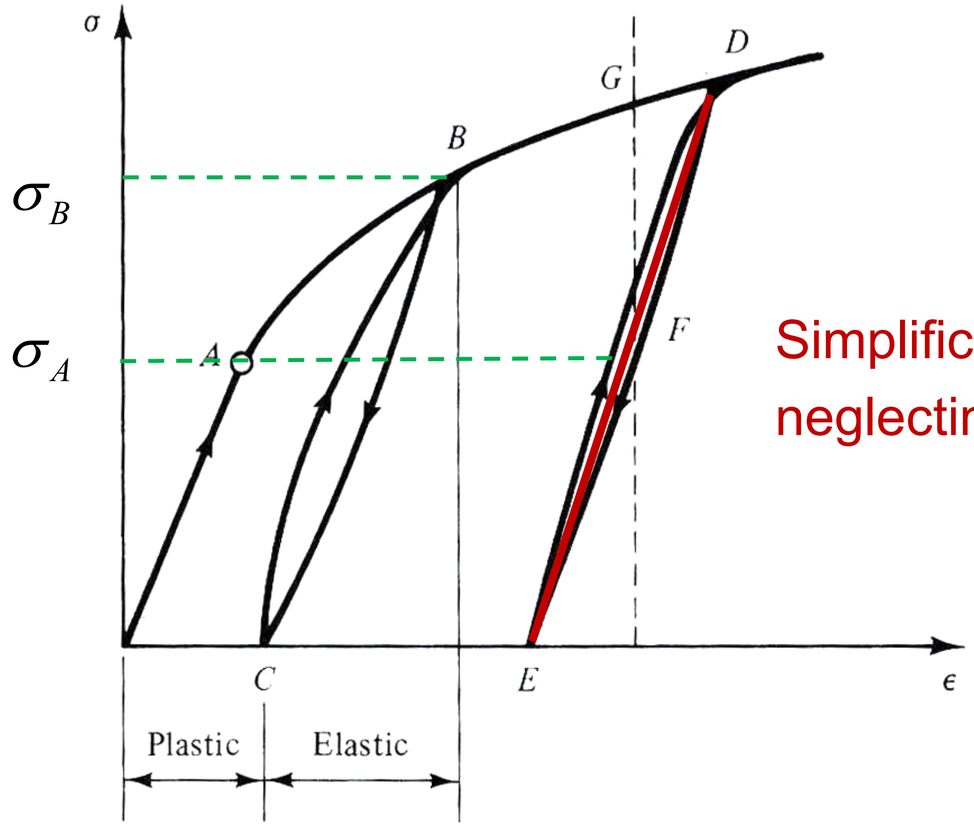
Plastic deformation

Total Elastic Plastic

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p}$$

Strain state depends not only on the actual stress state but also on the stress history

- **"Memory"**



Simplification:
neglecting the hysteresis

(Desai & Siriwardane, 1984)

Hardening behaviour

HARDENING BEHAVIOR AND HARDENING RULE

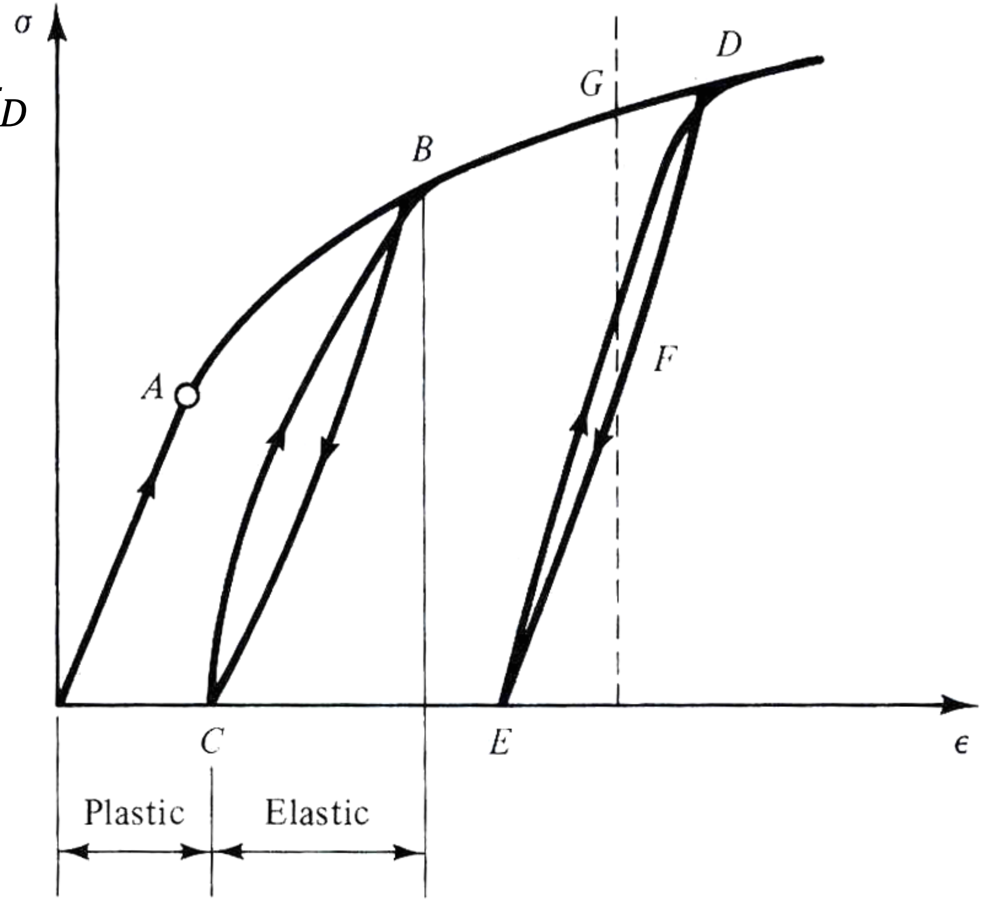
LOADING AND UNLOADING CONDITION

Hardening behaviour

Yield limit evolution

$F_{i+1}: \sigma_D$

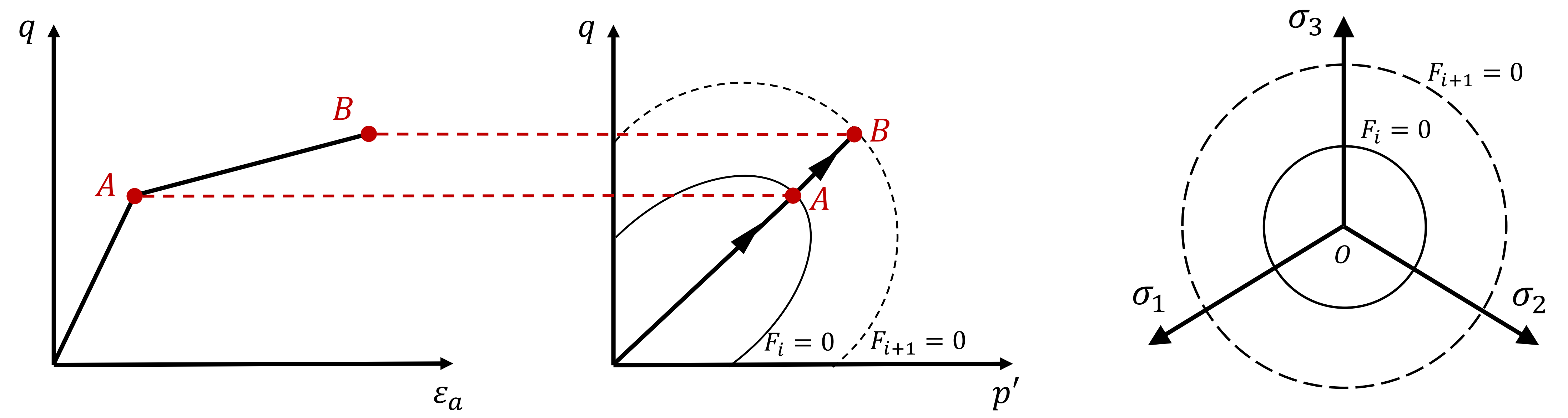
$F_i: \sigma_B$



(Desai & Siriwardane, 1984)

Isotropic hardening

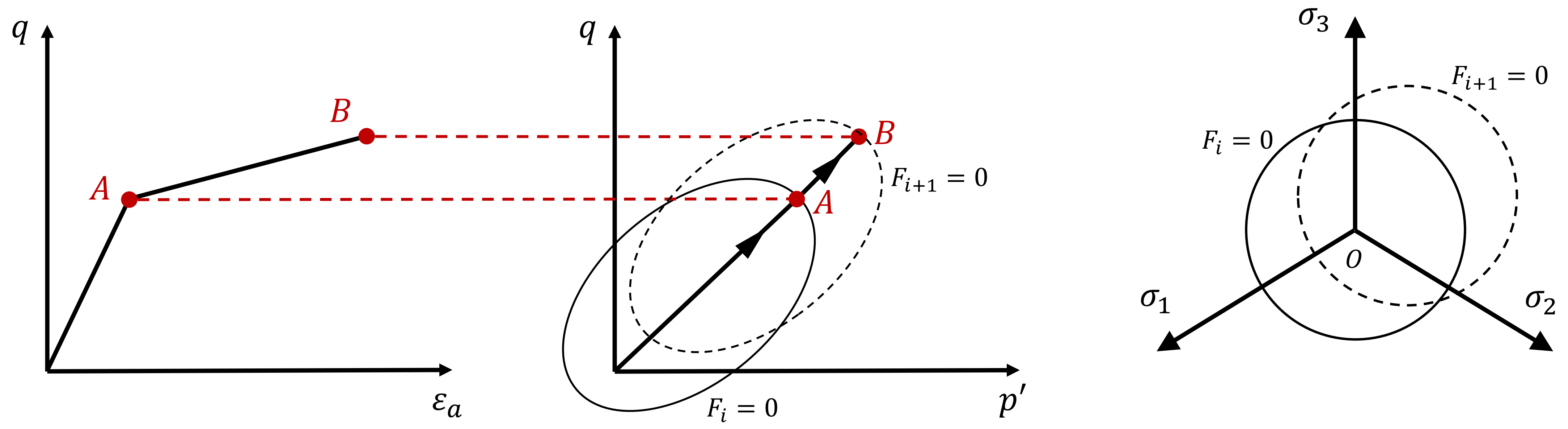
Change in the size of yield limit



Progressive yield surfaces

Kinematic hardening

Change in the position of yield limit



Yield surface remains the same shape and size → rigid transformation

Hardening

- **Yield function**

$F = F(\sigma_{ij}, p_k)$ p_k = geometric parameters

- **Plastic strain hardening**

The evolution of the geometrical parameters is linked to the evolution of plastic strains:

$$p_k = p_k(\varepsilon_{ij}^p) \longrightarrow F = F(\sigma_{ij}, \varepsilon_{ij}^p)$$

Many models use the volumetric plastic strain as hardening (or history) variable (volumetric hardening plasticity)

$$F = F(\sigma_{ij}, \varepsilon_v^p)$$

- **Hardening law**

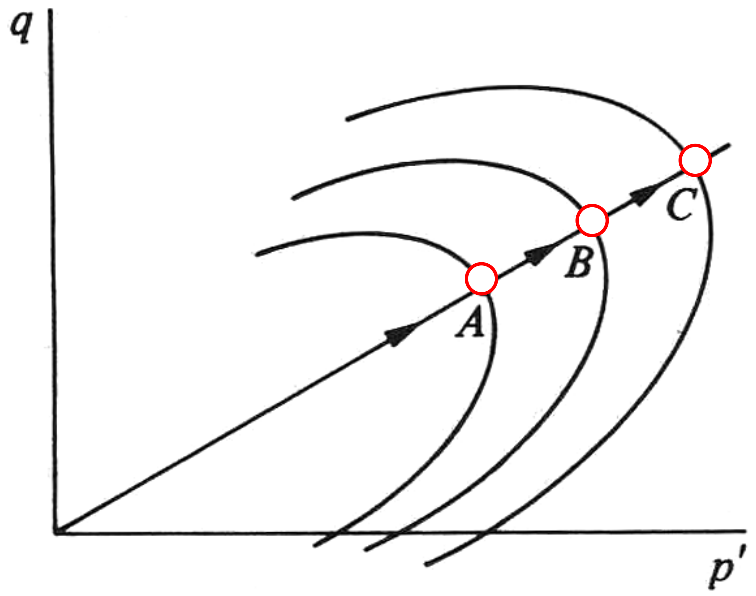
It expresses the link between changes in p_k and changes in plastic strain invariants (i.e. volumetric plastic strains)

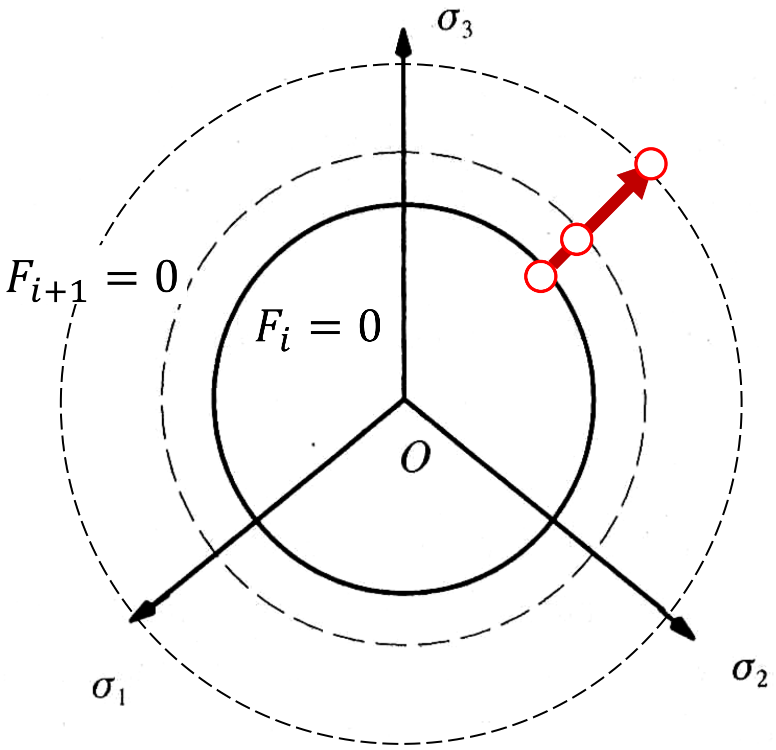
$$\delta p_k = function(\varepsilon_{ij}^p)$$

Consistency conditions

Consistency condition

During plasticity, stress state always stays on the yield surface





Consistency condition

Stress state always on the yield surface

$$dF = \left.\frac{\partial F}{\partial \sigma_i}\right|_{p_k} \delta\sigma_i + \left.\frac{\partial F}{\partial p_k}\right|_{\sigma_i} \delta p_k = 0$$

Plastic strain hardening

$$dF = \left.\frac{\partial F}{\partial \sigma_i}\right|_{p_k} \delta\sigma_i + \left.\frac{\partial F}{\partial p_k}\right|_{\sigma_i} \frac{\partial p_k}{\partial \varepsilon_i^p} \delta\varepsilon_i^p = 0$$

From the hardening rule

Hardening elasto-plastic stress-strain relationship

DERIVATION OF STRESS-STRAIN RELATIONSHIP

Derivation of stress-strain relationship

Objective: General incremental stress-strain relationship

1. Using elastic constitutive relation $\delta\sigma_i = D^e_{ij}\delta\varepsilon^e_j = D^e_{ij}\left(\delta\varepsilon_j - \delta\varepsilon^p_j\right)$

2. Using flow rule $\delta\varepsilon^p_j = \mu\dfrac{\partial g}{\partial\sigma_j}$

3. Replacing plastic strain by flow rule in the elastic constitutive relation $\delta\sigma_i = D^e_{ij}(\delta\varepsilon_j - \mu\dfrac{\partial g}{\partial\sigma_j})$

4. Using consistency equation $dF = \left.\dfrac{\partial F}{\partial\sigma_i}\right|_{p_k}\delta\sigma_i + \left.\dfrac{\partial F}{\partial p_k}\right|_{\sigma_i}\dfrac{\partial p_k}{\partial\varepsilon^p_j}\delta\varepsilon^p_i = \left.\dfrac{\partial F}{\partial\sigma_i}\right|_{p_k}\delta\sigma_i + \mu\left.\dfrac{\partial F}{\partial p_k}\right|_{\sigma_i}\dfrac{\partial p_k}{\partial\varepsilon^p_j}\dfrac{\partial g}{\partial\sigma_j} = 0$

Derivation of stress-strain relationship

Obtaining the plastic multiplier from the consistency equation:

$$\mu = -\frac{1}{\frac{\delta F}{\delta p_k}\frac{\delta p_k}{\delta \varepsilon_j^p}\frac{\delta g}{\delta \sigma_j}}\frac{\partial F}{\partial \sigma_i}\delta\sigma_i = \frac{1}{H}\frac{\partial F}{\partial \sigma_i}\delta\sigma_i \qquad \text{with} \qquad H = -\frac{\delta F}{\delta p_k}\frac{\delta p_k}{\delta \varepsilon_j^p}\frac{\delta g}{\delta \sigma_j}$$

Plastic modulus

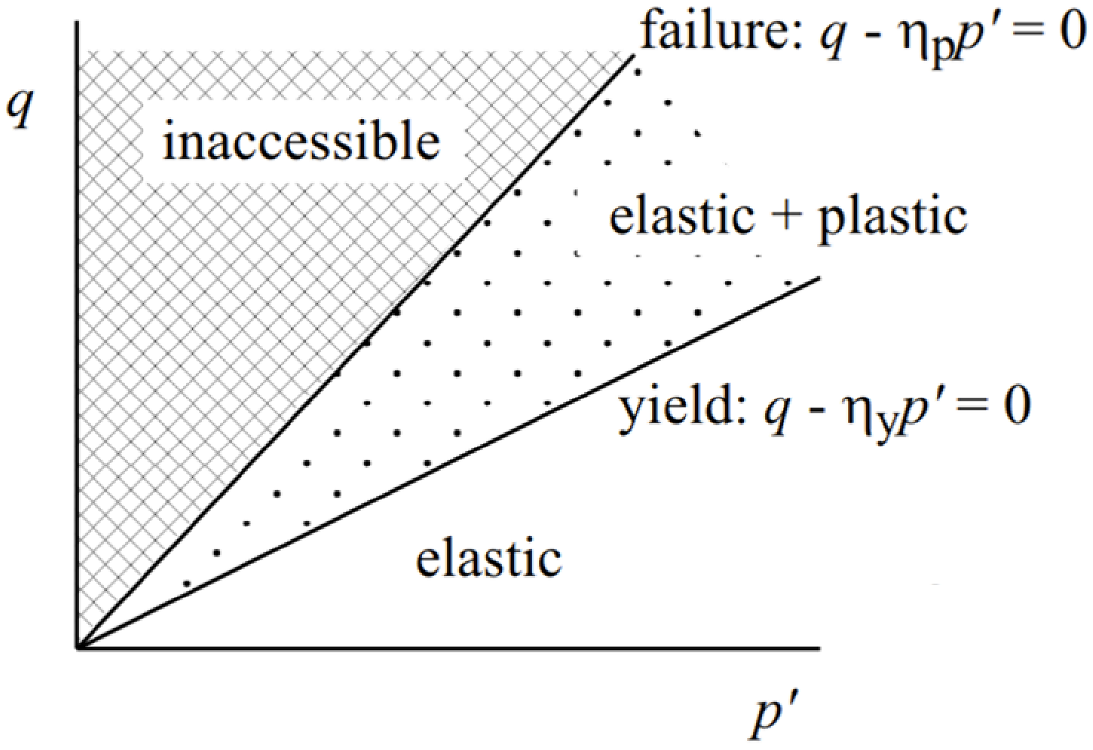
General incremental stress-strain relationship (Plastic strain isotropic hardening elasto-plasticity)

$$\delta\sigma_i = \underbrace{\left(D_{ij}^e - \frac{D_{ik}^e \frac{\partial g}{\partial \sigma_l}\frac{\partial F}{\partial \sigma_k} D_{kl}^e}{H + \frac{\partial F}{\partial \sigma_i} D_{ij}^e \frac{\partial g}{\partial \sigma_j}}\right)}_{D_{ij}^{ep}}\delta\varepsilon_j$$

Extended Mohr-Coulomb model (EMC)

EMC

- Hardening version of the perfectly plastic Mohr-Coulomb model
- Yield surface expands with deviatoric plastic strain accumulation
- Suitable for sandy materials where rearrangement of particles dominates the response
- It is convenient to refer to the stress ratio $\eta = q/p'$



Wood, 2004

EMC

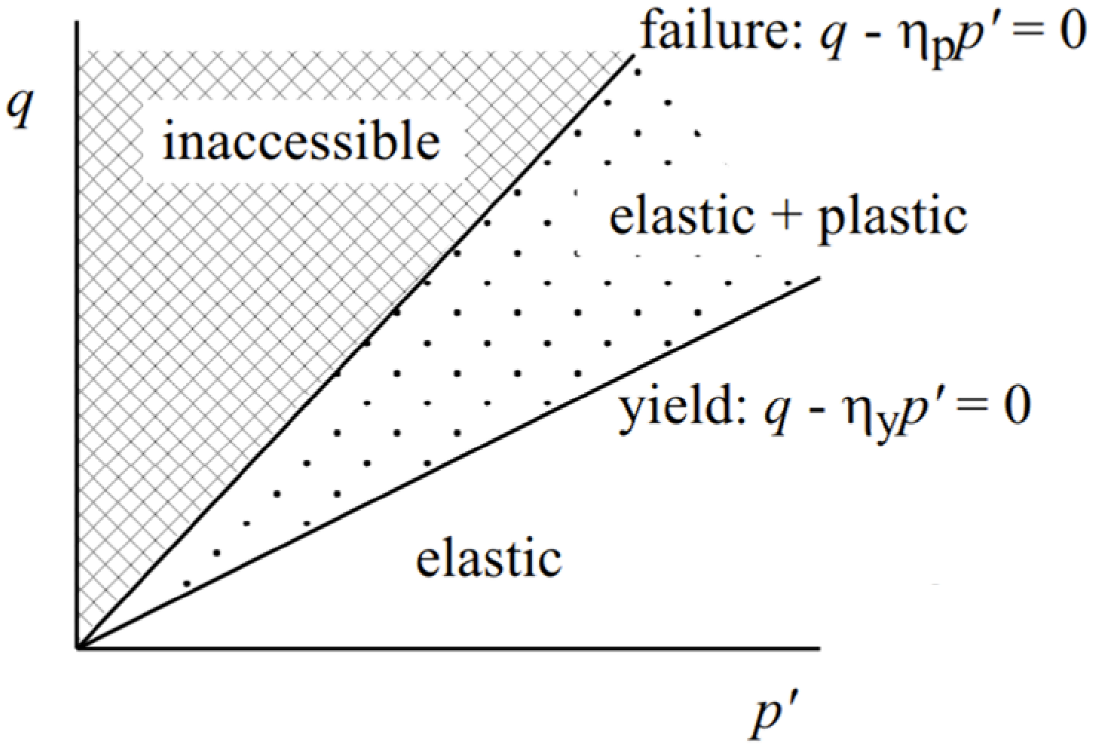
- The elastic component is isotropic and linear

$$\begin{bmatrix}\delta\varepsilon_v^e \\ \delta\varepsilon_d^e\end{bmatrix} = \begin{bmatrix}1/K & 0 \\ 0 & 1/3G\end{bmatrix}\begin{bmatrix}\delta p' \\ \delta q\end{bmatrix}$$

- The yield surface in the q, p' plane is a line inclined by η_y

$$F = F(\sigma_{ij}, p_k) = q - \eta_y p'$$

η_y is a hardening parameter which indicates the current size of the yield surface. It is allowed to expand progressively until it reaches a limiting value η_p



Wood, 2004

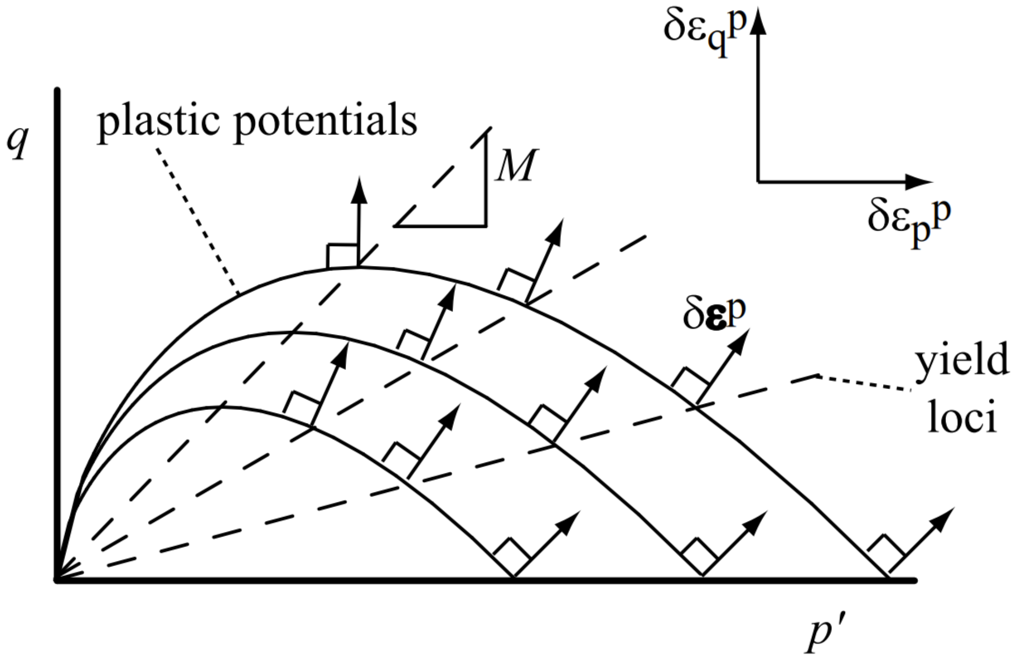
EMC

- Plastic Potential (non-associated flow rule)

Plastic potential surface $$g = q - Mp' \ln \frac{p'_r}{p'} = 0$$

A new parameter M is introduced

Volumetric and deviatoric plastic strain
$$\delta\varepsilon_v^p = \mu \frac{\partial g}{\partial p'} = \mu(M - \eta_y)$$
$$\delta\varepsilon_d^p = \mu \frac{\partial g}{\partial q} = \mu$$



Wood, 2004

Inclination of the plastic strain increment vector $$\frac{\delta\varepsilon_v^p}{\delta\varepsilon_d^p} = M - \frac{q}{p'} = M - \eta_y$$

M is the value of the stress ratio η at which shearing continues without further accumulation of volumetric plastic strain

EMC

Hardening Rule

- It defines how the elastic domain evolves with the cumulation of plastic strains

- In the EMC model, an hyperbolic relationship between stress ratio and deviatoric plastic strains is used:

$$\frac{\eta_y}{\eta_p} = \frac{\varepsilon_d^p}{a + \varepsilon_d^p}$$

  η_p limits the value of the stress ratio and a is a material parameter

- In incremental form:

$$\delta\eta_y = \frac{\left(\eta_p - \eta_y\right)^2}{a\eta_p}\delta\varepsilon_d^p$$

- Only the deviatoric plastic strain is responsible for the changes of the elastic domain

$$\begin{pmatrix} \partial\eta_y/\partial\varepsilon_v^p \\ \partial\eta_y/\partial\varepsilon_d^p \end{pmatrix} = \begin{pmatrix} 0 \\ \left(\eta_p - \eta_y\right)^2/a\eta_p \end{pmatrix}$$

EMC

Elasto-Plastic stiffness relationship

Combining all the ingredients, the complete elasto-plastic stiffness relationship is the following:

$$\begin{pmatrix} \delta \mathrm{p}' \\ \delta q \end{pmatrix} = \left[\begin{pmatrix} K & 0 \\ 0 & 3G \end{pmatrix} - \frac{\begin{pmatrix} -K^2\eta_y(M-\eta_y) & 3GK(M-\eta_y) \\ -3GK\eta_y & 9G^2 \end{pmatrix}}{3G - K\eta_y(M-\eta_y) + p'(\eta_p-\eta_y)^2/(a\eta_p)} \right] \begin{pmatrix} \delta\varepsilon_v \\ \delta\varepsilon_d \end{pmatrix}$$

Elastic part

Plastic part

EMC – example of application

Drained CTC with constant mean effective stress

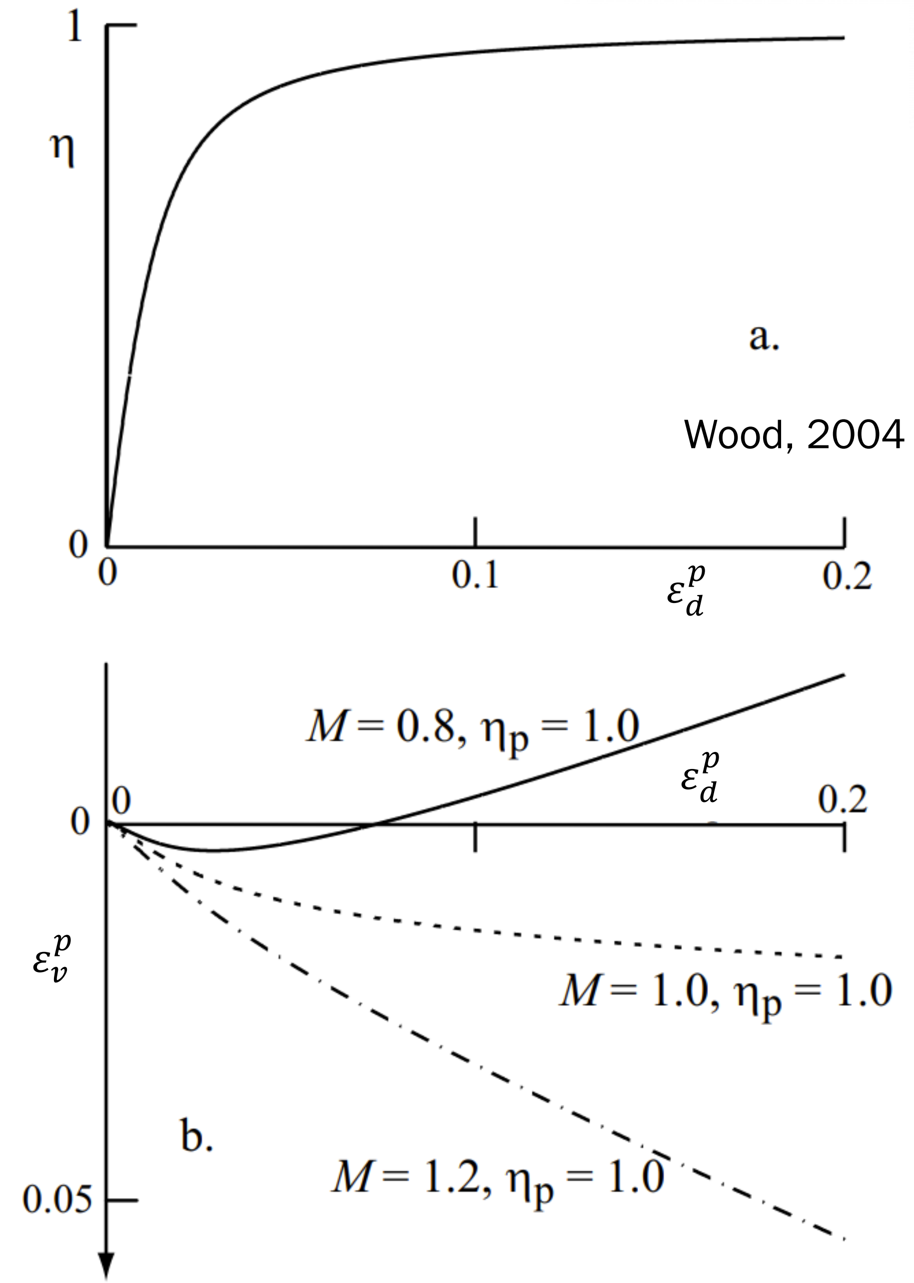
- No change in p' implies no generation of volumetric elastic strain. Total volumetric strain is equal to the plastic volumetric strain

- We can explore how the value of M influences the volumetric behaviour

  If $\eta_p > M$: volumetric compression followed by expansion

  If $\eta_p = M$: volumetric compression until constant volume shearing is reached

  If $\eta_p < M$: volumetric compression until failure



Summary

Summary

Ingredients of any elasto-plastic hardening model

(i) **Elastic properties:** to describe the recoverable deformation in elastic domain;

(ii) **Yield function:** to determine the limit when the occurrence plastic deformations starts;

(iii) **Plastic potential**: to understand the mechanism of plastic deformation and direction of plastic strains;

(iv) **Hardening rule:** to determine the magnitude of plastic deformation and evolution of yield surface.

Thank you for your attention